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A Fatou theorem for α-harmonic functions

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Abstract

We study functions which are harmonic in the upper half space with respect to $(-\Delta)^{\alpha/2}$, $0 < \alpha < 2$. We prove a Fatou theorem when the boundary function is L^p-Hölder continuous of order β and $\beta p > 1$. We give examples to show this condition is sharp.

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1. Introduction

A function is α-harmonic if it is harmonic with respect to a symmetric stable process of order α, $\alpha \in (0, 2)$, or equivalently, if it is harmonic with respect to the operator $(-\Delta)^{\alpha/2}$. It is now known that many potential-theoretic properties related to the Laplacian also hold for α-harmonic functions. As examples, consider the boundary Harnack principle, the characterization of the Martin boundary in a Lipschitz domain, intrinsic ultracontractivity, etc. See [2–13,15,17]. What is frequently the case is that while the proofs are quite different, the analogues for the α-harmonic case hold under less restrictive conditions than the corresponding theorems for harmonic functions. For example, the boundary Harnack principle for α-harmonic functions holds in bounded domains, whereas for the Laplacian more regularity is needed for the domain.

In this paper we consider the Fatou theorem. The classical Fatou theorem says that nonnegative harmonic functions in a ball converge nontangentiallly almost everywhere.

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The precise analogue of this for α-harmonic functions is *not* true. There are bounded functions which are α-harmonic in the upper half space where almost everywhere convergence fails. In contrast to the properties mentioned above, the correct theorem for the α-harmonic case requires more restrictive hypotheses. Some regularity of the boundary function is needed. Loosely speaking, the boundary function (which must be defined on the entire lower half space, because $(-\Delta)^{\alpha/2}$ is a non-local operator) must be L^p-Hölder continuous of order β for some β, p with $\beta p > 1$. (Recall that L^p-Hölder continuous functions need not be continuous.) Moreover the $\beta p > 1$ condition is sharp.

Let us now turn to a precise description of our results. Let $(X_t, \mathbb{P}^x)$ be a symmetric stable process of index α for some $\alpha \in (0,2)$. Its characteristic function has the form

$$\mathbb{E}^0 \mathrm{e}^{i\xi X_t} = \mathrm{e}^{-t|\xi|^\alpha}, \quad \xi \in \mathbb{R}^d,\ t \geqslant 0.$$

For a Borel set $A \subset \mathbb{R}^d$, we define $\tau_A = \inf\{t \geqslant 0\colon\ X_t \in A^c\}$, the first exit time of A. Let u be a Borel measurable function on $\mathbb{R}^d$ which is bounded from below. We say that u is α-harmonic in an open set $D \subset \mathbb{R}^d$ if

$$u(x) = \mathbb{E}^x u(X_{\tau_B}), \quad x \in B,$$

for every bounded open set B whose closure $\overline{B}$ is a subset of D. We say that u is a regular α-harmonic function in D if

$$u(x) = \mathbb{E}^x u(X_{\tau_D}), \quad x \in D.$$

Note that regular α-harmonic functions are α-harmonic by the strong Markov property of X_t. One can also give an analytic definition (i.e., without using probability) for α-harmonic and regular α-harmonic; for the latter see the end of Section 2.

For points in $\mathbb{R}^d$ we write $x = (\tilde{x}, x_d)$, where $\tilde{x} \in \mathbb{R}^{d-1}$. Let

$$H = \left\{x \in \mathbb{R}^d\colon\ x = (\tilde{x}, x_d),\ x_d > 0\right\}$$

be the upper half space. We write ∂H for $\{(\tilde{x}, 0)\colon\ \tilde{x} \in \mathbb{R}^{d-1}\}$. If f is a measurable function on H^c, define

$$u_f(x) = \mathbb{E}^x f(X_{\tau_H}), \quad x \in H. \tag{1.1}$$

Then u_f will be regular α-harmonic in H. For stable symmetric processes, the support of the distribution of X_{τ_H} is all of H^c and not just ∂H; see Proposition 2.1.

Let $\Lambda_\beta^{p,\infty}(\mathbb{R}^d)$ be the space of all functions f in $L^p(\mathbb{R}^d)$, where $\beta > 0$ and $p \geqslant 1$, for which the norm

$$\|f\|_p + \sup_{|t|>0} \frac{\|f(x+t) - f(x)\|_p}{|t|^\beta} \tag{1.2}$$

is finite. This is the space of L^p-Hölder continuous functions of order β. See Stein [16] for further information about this space. We will say that a function f defined on H^c is in $\Lambda_\beta^{p,\infty}(H^c)$ if there exists a function $\bar{f} \in \Lambda_\beta^{p,\infty}(\mathbb{R}^d)$ such that f is the restriction of $\bar{f}$ to H^c. We define $\|f\|_{\Lambda_\beta^{p,\infty}(H^c)} = \|\bar{f}\|_{\Lambda_\beta^{p,\infty}(\mathbb{R}^d)}$.

Let $\Theta > 0$ be fixed and let

$$\Gamma_{\tilde{x}} = \left\{y\colon\ y_d > 0,\ |\tilde{y}| < \Theta y_d\right\}$$

be the cone with vertex at $\tilde{x}$. We say that a function g converges nontangentially at $\tilde{x} \in \partial H$ if

$$\lim_{z \in \Gamma_{\tilde{x}},\ z \to \tilde{x}} g(z)$$

exists.

Our main result is the following,

Theorem 1.1. *Suppose $f \in \Lambda_\beta^{p,\infty}(H^c) \cap L^{p_0}$ for some $\beta \in (0,1)$, $p \in (1,\infty]$, and $p_0 \in (1,\infty]$. Let $u = u_f$ be the regular α-harmonic function in H associated with f defined by* (1.1). *If*

(a) *f is locally bounded,*
(b) *$p_0 > 1/(1-\alpha/2)$, and*
(c) *$\beta p > 1$,*

then the nontangential limit of u exists for almost every $\tilde{x} \in \partial H$.

The conditions in this theorem are sharp. It is easy to check that (a) and (b) are needed, but we also have

Theorem 1.2. *For each $\beta \in (0,1)$ and $p \in (1, 1/\beta]$, there exist $f \in \Lambda_\beta^{p,\infty}(H^c)$ and $A \subset \partial H$ such that f is bounded and has compact support, A has positive $(d-1)$-Lebesgue measure, and u_f does not converge nontangentially at any point of A.*

The paper is organized as follows. In Section 2 we compute the Poisson kernel for the upper half space. In Section 3 we introduce a type of maximal function and establish an estimate for it. In Section 4 we prove Theorem 1.1 and we give our examples in Section 5.

We use the letter c, with or without subscripts, to denote positive finite constants whose exact value is unimportant. Let $B(x_0, r) = \{x \in \mathbb{R}^d\colon |x - x_0| < r\}$ be the open ball centered at x_0 with radius r. Given a Borel subset D of $\mathbb{R}^d$, let $|D|$ denote the Lebesgue measure of D.

2. Poisson kernel

It is known (see [14, pp. 121–122]) that the distribution of $X_{\tau_{B(0,r)}}$ under $\mathbb{P}^x$ has a density with respect to Lebesgue measure on $\mathbb{R}^d$ and the density function $P_r(x,\cdot)$, also known as the Poisson kernel, is explicitly given by the formula

$$P_r(x,y) = c(\alpha,d)\left[\frac{r^2 - |x|^2}{|y|^2 - r^2}\right]^{\alpha/2} |x-y|^{-d}$$

when $x \in B(0,r)$ and $y \notin B(0,r)$. Here $c(\alpha,d) = \Gamma(d/2)\pi^{-d/2-1}\sin(\pi\alpha/2)$. Just as in the case of the Laplacian, this Poisson kernel representation allows one to easily obtain a Harnack inequality for nonnegative α-harmonic functions.

Proposition 2.1. *If $x \in H$, the distribution of X_{τ_H} under $\mathbb{P}^x$ has a density with respect to d-dimensional Lebesgue measure given by*

$$P_H(x, y) = c(\alpha, d)\left(\frac{x_d}{|y_d|}\right)^{\alpha/2} |x-y|^{-d}, \quad y_d < 0.$$

Proof. Let $B_n = B(ne_d, n)$, where $e_d = (\tilde{0}, 1)$. Note $\tau_{B_n} \uparrow \tau_H$. Since the process (X_t) is quasi-left continuous, we have

$$\lim_{n\to\infty} X_{\tau_{B_n}} = X_{\tau_H} \quad \text{a.s.;}$$

see [1, pp. 17–18, 45, and 51].

So $\mathbb{E}^x f(X_{\tau_{B_n}}) \to \mathbb{E}^x f(X_{\tau_H})$ as $n \to \infty$ for any bounded and continuous function f with compact support. For such an f

$$\mathbb{E}^x f(X_{\tau_{B_n}}) = c(\alpha, d) \int_{B_n^c} \left(\frac{x_d - \frac{|x|^2}{2n}}{\frac{|y|^2}{2n} - y_d}\right)^{\alpha/2} |x-y|^{-d} f(y)\, dy. \tag{2.1}$$

If f is continuous with compact support and $x \in H$ is fixed, then for y in B_n^c, $|x-y|^{-d}$ is bounded by a finite number independent of n. It is easy to see that if S is the support of f and ε is sufficiently small, then

$$\left(\frac{x_d - \frac{|x|^2}{2n}}{\frac{|y|^2}{2n} - y_d}\right)^{(1+\varepsilon)\alpha/2} \chi_{S\cap B_n^c}(y)$$

has an integral over y that is bounded by a finite number that does not depend on n. Therefore the integrand in (2.1) is uniformly integrable with respect to the finite measure $\chi_S(y)\, dy$ and the limit of the right-hand side of (2.1) as $n \to \infty$ is

$$c(\alpha, d) \int_{H^c} \left(\frac{x_d}{|y_d|}\right)^{\alpha/2} |x-y|^{-d} f(y)\, dy.$$

Our result now follows. $\square$

One can now say what it means for a function u_f defined in H to be regular α-harmonic in H without using probability. u_f will be regular α-harmonic in H if there exists f defined on H^c such that

$$u_f(x) = \int_{H^c} P_H(x, y) f(y)\, dy$$

for all $x \in H$.

3. Maximal functions

Suppose $f \in \Lambda_\beta^{p,\infty}(H^c)$ and suppose also for now that the support of f is a bounded set. For i an integer let $A_i(\tilde{x})$ be the cube in $\mathbb{R}^d$ with center $(\tilde{x}, -2^{-i})$ and side length 2^{-i}.

We will write A_i for $A_i(\tilde{0})$. We will use $\tilde{A}_i(\tilde{x})$ to denote the $(d-1)$-dimensional cube with center $\tilde{x}$ and side length 2^{-i}. Set

$$\mathcal{B}_i = \{\tilde{x} \in \mathbb{R}^{d-1}: \text{ each coordinate is a multiple of } 2^{-i}\}.$$

Let us define for $i \geqslant 0$

$$F_i(\tilde{x}) = \frac{1}{|A_i(\tilde{x})|} \int_{A_i(\tilde{x})} |f(y)|\,dy, \qquad G_i(\tilde{x}) = \sup_{\tilde{y} \in \mathcal{B}_i,\ |\tilde{y}| \leqslant (i+1)2^{-i}} F_i(\tilde{x} + \tilde{y}).$$

Note that the integral defining F_i is a d-dimensional one. Finally, define the maximal function $Mf : \partial H \to [0, \infty]$ by the following.

$$Mf(\tilde{x}) = \sup_{i \geqslant 0} G_i(\tilde{x}).$$

We use $\|\cdot\|_1$ to denote the L^1 norm of a function on ∂H with respect to $(d-1)$-dimensional Lebesgue measure.

Proposition 3.1. *Suppose $f \in \Lambda_\beta^{p,\infty}(H^c)$, $\beta p > 1$, $p > 1$, and the support of f is a bounded set. Let $S = B(0, J)$ be a ball with $J \geqslant 1$ containing the support of f. Then*

$$\|Mf(\tilde{x})\|_1 < c_1 J^{(d-1)(1-(1/p))} \|f\|_{\Lambda_\beta^{p,\infty}(H^c)}.$$

The constant c_1 depends on p and β. In particular, $Mf(\tilde{x})$ is finite a.e.

Proof. Since

$$|G_i(\tilde{x}) - G_{i+1}(\tilde{x})| \leqslant \sup_{\tilde{y}, \tilde{z} \in \mathcal{B}_{i+1},\ |\tilde{y}| \vee |\tilde{z}| \leqslant 2(i+2)2^{-i}} |F_i(\tilde{x} + \tilde{y}) - F_{i+1}(\tilde{x} + \tilde{z})|,$$

we have

$$\|G_i(\tilde{x}) - G_{i+1}(\tilde{x})\|_1 \leqslant \sup_{\tilde{w} \in \mathcal{B}_{i+1},\ |\tilde{w}| \leqslant 4(i+2)2^{-i}} \|F_i(\tilde{x}) - F_{i+1}(\tilde{x} + \tilde{w})\|_1.$$

Let t_j, $j = 1, \ldots, 2^d$, be points such that $A_i = \bigcup_{j=1}^{2^d} (t_j + A_{i+1})$; we can find a constant c_2 not depending on i such that $|t_j| \leqslant c_2 2^{-i}$ for $j = 1, \ldots, 2^d$. Fix a $\tilde{w} \in \mathcal{B}_{i+1}$ with $|\tilde{w}| \leqslant 4(i+2)2^{-i}$. Note that

$$\begin{aligned}
&F_i(\tilde{x}) - F_{i+1}(\tilde{x} + \tilde{w}) \\
&\quad = \frac{1}{|A_i(\tilde{x})|} \int_{A_i(\tilde{x})} |f(y)|\,dy - \frac{1}{|A_{i+1}(\tilde{x} + \tilde{w})|} \int_{A_{i+1}(\tilde{x}+\tilde{w})} |f(y)|\,dy \\
&\quad = \sum_{j=1}^{2^d} \frac{1}{|A_i|} \int_{A_{i+1}} \big[|f((\tilde{x}, 0) + y + t_j)| - |f((\tilde{x} + \tilde{w}, 0) + y)|\big]\,dy \\
&\quad = \sum_{j=1}^{2^d} 2^{id} \int_{A_{i+1}} \big[|f((\tilde{x} + \tilde{y}, y_d) + t_j)| - |f(\tilde{x} + \tilde{y} + \tilde{w}, y_d)|\big]\,d\tilde{y}\,dy_d.
\end{aligned} \tag{3.1}$$

Then integrating (3.1) gives

$$\begin{aligned}
&\int\limits_{\mathbb{R}^{d-1}} \big|F_i(\tilde{x}) - F_{i+1}(\tilde{x}+\tilde{w})\big|\,d\tilde{x} \\
&\quad\leqslant \sum_{j=1}^{2^d} 2^{id} \int\limits_{\mathbb{R}^{d-1}} \Bigg(\int\limits_{A_{i+1}} \big|f\big((\tilde{x}+\tilde{y}, y_d)+t_j\big) - f(\tilde{x}+\tilde{y}+\tilde{w}, y_d)\big|\,dy\Bigg)\,d\tilde{x} \\
&\quad= \sum_{j=1}^{2^d} 2^{id} \int\limits_{\mathbb{R}^{d-1}} \int\limits_{-3\cdot 2^{-i-2}}^{-2^{-i-2}} \int\limits_{\tilde{A}_{i+1}} \big|f\big((\tilde{x}+\tilde{y}, y_d)+t_j\big) - f(\tilde{x}+\tilde{y}+\tilde{w}, y_d)\big|\,d\tilde{y}\,dy_d\,d\tilde{x}.
\end{aligned}$$

With the change of variable $z = \tilde{x}+\tilde{y}$ and the Fubini theorem, the last expression can be written as

$$\begin{aligned}
&\sum_{j=1}^{2^d} 2^{id} \int\limits_{-3\cdot 2^{-i-2}}^{-2^{-i-2}} \int\limits_{\tilde{A}_{i+1}} \int\limits_{\mathbb{R}^{d-1}} \big|f\big((z, y_d)+t_j\big) - f(z+\tilde{w}, y_d)\big|\,dz\,d\tilde{y}\,dy_d \\
&\quad\leqslant \sum_{j=1}^{2^d} 2^{id} \int\limits_{-2^{-i}}^{-2^{-i-2}} |\tilde{A}_{i+1}| \int\limits_{\mathbb{R}^{d-1}} \big|f\big((z, y_d)+t_j\big) - f\big((z, y_d)+(\tilde{w}, 0)\big)\big|\,dz\,dy_d \\
&\quad= \sum_{j=1}^{2^d} 2^{i} \int\limits_{-2^{-i}}^{-2^{-i-2}} \int\limits_{\mathbb{R}^{d-1}} \big|f\big((z, y_d)+t_j\big) - f\big((z, y_d)+(\tilde{w}, 0)\big)\big|\,dz\,dy_d.
\end{aligned}$$

Let $w = (\tilde{w}, 0)$. Then we have

$$\begin{aligned}
&\int\limits_{\mathbb{R}^{d-1}} \big|F_i(\tilde{x}) - F_{i+1}(\tilde{x}+\tilde{w})\big|\,d\tilde{x} \\
&\quad\leqslant \sum_{j=1}^{2^d} 2^{i} \int\limits_{-2^{-i}}^{-2^{-i-2}} \int\limits_{\mathbb{R}^{d-1}} \big|f\big((z, y_d)+t_j\big) - f\big((z, y_d)+w\big)\big|\,dz\,dy_d \\
&\quad\leqslant \sum_{j=1}^{2^d} 2^{i} \Bigg(\int\limits_{-2^{-i}}^{-2^{-i-2}} \int\limits_{\mathbb{R}^{d-1}} \big|f\big((z, y_d)+t_j\big) - f\big((z, y_d)+w\big)\big|^p\,dz\,dy_d \Bigg)^{1/p} \\
&\qquad\times \Bigg(\iint \chi_{(S\cap(\mathbb{R}^{d-1}\times[-2^{-i}, -2^{-i-2}]))}\,dz\,dy_d \Bigg)^{1/q} \\
&\quad\leqslant c_3 J^{(d-1)/q} \sum_{j=1}^{2^d} 2^{i} |t_j - w|^{\beta} \|f\|_{\Lambda_{\beta}^{p,\infty}(\mathbb{R}^d)} 2^{-i/q} \qquad (3.2)
\end{aligned}$$

by Hölder's inequality, where $p^{-1}+q^{-1}=1$. Since $|t_j-w|\leqslant c_4 2^{-i}(i+2)$, for $j=1,\dots,2^d$, $\tilde w\in\mathcal{B}_{i+1}$, and $|\tilde w|\leqslant 4(i+2)2^{-i}$, using (3.2) we have

$$\begin{aligned}\bigl\|G_i(\tilde x)-G_{i+1}(\tilde x)\bigr\|_1&\leqslant c_5 J^{(d-1)/q}2^{i(1-1/q)}\bigl(2^{-i}(i+2)\bigr)^{\beta}\|f\|_{\Lambda_\beta^{p,\infty}(\mathbb{R}^d)}\\&\leqslant c_6 J^{(d-1)/q}(i+2)^{\beta}2^{(-\beta+(1/p))i}\|f\|_{\Lambda_\beta^{p,\infty}(\mathbb{R}^d)}.\end{aligned}\tag{3.3}$$

To prove $\|Mf(\tilde x)\|_1<c_1J^{(d-1)(1-(1/p))}\|f\|_{\Lambda_\beta^{p,\infty}(H^c)}$, note that

$$\sup_i\bigl|G_i(\tilde x)\bigr|\leqslant\bigl|G_0(\tilde x)\bigr|+\sum_{i=1}^{\infty}\bigl|G_i(\tilde x)-G_{i-1}(\tilde x)\bigr|.$$

Clearly $\|G_0\|_1<c_7\|f\|_p$ because f is in L^p and has compact support. Hence by (3.3)

$$\begin{aligned}&\bigl\|Mf(\tilde x)\bigr\|_1\\&\quad=\bigl\|\sup_i G_i(\tilde x)\bigr\|_1\\&\quad\leqslant\bigl\|G_0(\tilde x)\bigr\|_1+\sum_{i=1}^{\infty}\bigl\|G_i(\tilde x)-G_{i-1}(\tilde x)\bigr\|_1\\&\quad\leqslant c_8J^{(d-1)/q}\|f\|_{\Lambda_\beta^{p,\infty}(\mathbb{R}^d)}+c_8J^{(d-1)/q}\|f\|_{\Lambda_\beta^{p,\infty}(\mathbb{R}^d)}\sum_{i=1}^{\infty}(i+2)^{\beta}2^{((1/p)-\beta)i}\\&\quad<c_9J^{(d-1)/q}\|f\|_{\Lambda_\beta^{p,\infty}(\mathbb{R}^d)},\end{aligned}\tag{3.4}$$

if $\beta p>1$. □

4. The Fatou theorem

In this section we prove Theorem 1.1. We start with the following lemma.

Lemma 4.1. *Suppose f is bounded, has support contained in $B(0,J)$ for some $J>1$, and $f\in\Lambda_\beta^{p,\infty}(\mathbb{R}^d)$ for some $p<\infty$. Let $\beta'<\beta$ and $\delta>0$. We can write $f=g+h$, where g is continuous with support in $B(0,J+1)$, $\|h\|_\infty\leqslant 2\|f\|_\infty$, and $\|h\|_{\Lambda_{\beta'}^{p,\infty}}<\delta$.*

Proof. Let $\psi(x)$ be a nonnegative continuous function with support in $B(0,1)$ such that $\int\psi(x)\,dx=1$. Set $\psi_\varepsilon(x)=\varepsilon^{-d}\psi(x/\varepsilon)$, $g_\varepsilon=f*\psi_\varepsilon$, and $h_\varepsilon=f-g_\varepsilon$. No matter what ε is, we have

$$\|h_\varepsilon\|_\infty\leqslant\|f\|_\infty+\|f*\psi_\varepsilon\|_\infty\leqslant 2\|f\|_\infty,$$

and as long as $\varepsilon\leqslant 1$, then g_ε will be continuous with support in $B(0,J+1)$.

Choose r such that $2\|f\|_{\Lambda_\beta^{p,\infty}(\mathbb{R}^d)}r^{\beta-\beta'}<\delta/2$. Since $f\in L^p$, then $\|h_\varepsilon\|_p\to 0$ as $\varepsilon\to 0$. Choose $\varepsilon<1$ small enough so that

$$\|h_\varepsilon\|_p\leqslant\frac{\delta}{4}\bigl(1\wedge r^{\beta'}\bigr).$$

If $|t| \leqslant r$,

$$\begin{aligned}\big\|h_\varepsilon(x+t)-h_\varepsilon(x)\big\|_p &\leqslant \big\|f(x+t)-f(x)\big\|_p+\big\|f*\psi_\varepsilon(x+t)-f*\psi_\varepsilon(x)\big\|_p\\ &\leqslant 2\big\|f(x+t)-f(x)\big\|_p \leqslant 2\|f\|_{\Lambda_\beta^{p,\infty}(\mathbb{R}^d)}|t|^\beta\\ &\leqslant 2\|f\|_{\Lambda_\beta^{p,\infty}(\mathbb{R}^d)} r^{\beta-\beta'}|t|^{\beta'} < \frac{\delta}{2}|t|^{\beta'}.\end{aligned}$$

If $|t| > r$,

$$\begin{aligned}\big\|h_\varepsilon(x+t)-h_\varepsilon(x)\big\|_p &\leqslant \big\|h_\varepsilon(x+t)\big\|_p+\|h_\varepsilon\|_p\\ &< 2\left(\frac{\delta}{4}\right)r^{\beta'} \leqslant \frac{\delta}{2}|t|^{\beta'}.\end{aligned}$$

Take $g = g_\varepsilon$, $h = h_\varepsilon$. Hence $\|h\|_{\Lambda_{\beta'}^{p,\infty}} < \delta$. $\square$

The main step in the proof of Theorem 1.1 is the following proposition.

Proposition 4.2. *Suppose f is the restriction to H of a function $\bar{f} \in \Lambda_\beta^{p,\infty}(\mathbb{R}^d)$ which is bounded and which has support in $B(0, J)$ for some $J > 1$. Suppose $\beta p > 1$. Then u_f will converge nontangentially for almost every $\tilde{x} \in \partial H$.*

Proof. If $p = \infty$, then f is continuous and the result is easy, so we assume $p < \infty$. Let $\varepsilon > 0$, where we will specify the exact value later on. Let

$$T(a,b) = \bigcup_{i=-a}^{a} \bigcup_{\{t_j\in\mathcal{B}_i,\ |t_j|\leqslant b2^{-i}\}} A_i(\tilde{x}+t_j)$$

for a, b positive integers. Then $T(a,b) \uparrow \overline{H}^c$ as $a, b \to \infty$. So by dominated convergence

$$\int_{T(a,b)^c} P_H\big((\tilde{x},1),z\big)\,dz \to 0$$

as $a, b \to \infty$ and thus there exist a, b such that

$$\int_{T(a,b)^c} P_H\big((\tilde{x},1),z\big)\,dz < \varepsilon. \tag{4.1}$$

Let

$$T = \bigcup_{i=i_0-a-2}^{i_0+a+2} \bigcup_{\{t_j\in\mathcal{B}_i,\ |t_j|\leqslant(b+2)2^{-i}\}} A_i(\tilde{x}+t_j), \tag{4.2}$$

if $x_d \in [2^{-i_0-1}, 2^{-i_0})$ for some i_0. Then

$$\int_{T^c} P_H\big((\tilde{x}, x_d), w\big)\,dw = c(\alpha, d)\int_{T^c} \left(\frac{x_d}{|w_d|}\right)^{\alpha/2} \frac{1}{|(\tilde{w}, w_d) - x|^d}\,dw$$
$$\leqslant c(\alpha, d)\int_{T(a,b)^c} \left(\frac{1}{|z_d|}\right)^{\alpha/2} \frac{1}{|(\tilde{z}, z_d) - x|^d}\,dz$$
$$= \int_{T(a,b)^c} P_H\big((\tilde{x}, 1), z\big)\,dz, \tag{4.3}$$

by a change of variables. By (4.1) and (4.3), we have $\int_{T^c} P_H((\tilde{x}, x_d), z)\,dz < \varepsilon$. Now let h be a bounded function. Then we have

$$\left|\int_{T^c} h(z) P_H\big((\tilde{x}, x_d), z\big)\,dz\right| < \varepsilon \|h\|_\infty. \tag{4.4}$$

Let

$$B(\tilde{x}) = \bigcup_{i=1}^{\infty} \bigcup_{\{t_j \in \mathcal{B}_i,\ |t_j| \leqslant (i+1)2^{-i}\}} A_i(\tilde{x} + t_j).$$

By a change of variables, we have

$$\int_{B(\tilde{x})^c} P_H\big((\tilde{x}, x_d), w\big)\,dw = \int_{C(x_d)^c} P_H\big((\tilde{x}, 1), z\big)\,dz,$$

where $C(x_d)$ is the image of $B(\tilde{x})$ under the change of variables. Notice that $C(x_d) \uparrow \overline{H}^c$ as $x_d \to 0$, and so $\int_{C(x_d)^c} P_H((\tilde{x}, 1), z) \to 0$ as $x \to 0$ by dominated convergence. Hence there exists γ such that for $x_d < \gamma$ we have $\int_{B(\tilde{x})^c} P_H((\tilde{x}, x_d), w)\,dw < \varepsilon$, so

$$\left|\int_{B(\tilde{x})^c} h(z) P_H\big((\tilde{x}, x_d), z\big)\,dz\right| < \varepsilon \|h\|_\infty. \tag{4.5}$$

Suppose $x_d \in [2^{-i_0-1}, 2^{-i_0})$, $i_0 - a - 2 \leqslant i \leqslant i_0 + a + 2$, $t_j \in \mathcal{B}_i$, and $|t_j| \leqslant (b+2)2^{-i}$. If γ is small enough and $x_d < \gamma$, then i_0 will be large and then also $|t_j| \leqslant (i+1)2^{-i}$. For $z = (\tilde{z}, z_d) \in A_i(\tilde{x} + t_j)$ we see that $x_d/|z_d| \leqslant c_1 2^a$ and $|x - z|^{-d} \leqslant c_2(a, b)2^{id}$. Hence

$$\int_{A_i(\tilde{x}+t_j)} \big|h(z)\big| P_H(x, z)\,dz \leqslant c(\alpha, d)\int_{A_i(\tilde{x}+t_j)} \big|h(z)\big| \left(\frac{x_d}{|z_d|}\right)^{\alpha/2} |x - y|^{-d}\,dz$$
$$\leqslant c_3(a, b)2^{id} \int_{A_i(\tilde{x}+t_j)} \big|h(z)\big|\,dz$$
$$\leqslant c_3(a, b) G_i(\tilde{x}).$$

Summing over t_j and i,

$$\int_{T \cap B(\tilde{x})} \big|h(z)\big| P_H(x, z)\,dz \leqslant c_4(a, b) Mh(\tilde{x}). \tag{4.6}$$

Observe that

$$H^c = \big(T \cap B(\tilde{x})\big) \cup \big(T \cap B(\tilde{x})^c\big) \cup T^c.$$

So if $x_d < \gamma$ and $x_d \in [-2^{-i_0}, -2^{-i_0-1})$ for $i_0 \geqslant 1$,

$$\begin{aligned} u_h(\tilde{x}, x_d) &= \int h(z) P_H\big((\tilde{x}, x_d), z\big)\, dz \\ &= \int\limits_{T\cap B(\tilde{x})} h(z) P_H\big((\tilde{x}, x_d), z\big)\, dz + \int\limits_{T\cap B(\tilde{x})^c} h(z) P_H\big((\tilde{x}, x_d), z\big)\, dz \\ &\quad + \int\limits_{T^c} h(z) P_H\big((\tilde{x}, x_d), z\big)\, dz, \end{aligned}$$

which implies

$$\big|u_h(\tilde{x}, x_d)\big| \leqslant c_4(a,b) Mh(\tilde{x}) + 2\varepsilon \|h\|_\infty \tag{4.7}$$

by (4.4), (4.5), and (4.6).

Let $\eta > 0$. Choose $\beta' \in (0, \beta)$ such that $\beta' p > 1$. Let $\varepsilon = \eta / \|f\|_\infty$, choose a, b large so that (4.1) holds, and then set $\delta = \eta / (c_4(a,b)(J+1)^{(d-1)(1-(1/p))})$. Use Lemma 4.1 to write $\bar{f} = \bar{g} + \bar{h}$, where $\bar{g}$ is continuous with support in $B(0, J+1)$, $\|\bar{h}\|_\infty \leqslant 2\|f\|_\infty$, and $\|\bar{h}\|_{\Lambda^{p,\infty}_{\beta'}} < \delta$. Let g and h be the restrictions to H^c of $\bar{g}$ and $\bar{h}$, respectively.

Let us denote for each function k defined on H^c:

$$\Omega k(\tilde{x}) = \Big| \limsup_{z \in \Gamma_{\tilde{x}},\ z \to \tilde{x}} u_k(z) - \liminf_{z \in \Gamma_{\tilde{x}},\ z \to \tilde{x}} u_k(z) \Big|.$$

Then $\Omega f(\tilde{x}) \leqslant \Omega g(\tilde{x}) + \Omega h(\tilde{x})$. Since g is continuous with compact support, it is easy to see that $\Omega g(\tilde{x}) = 0$ for each $\tilde{x} \in \partial H$. We therefore have $\Omega f(\tilde{x}) \leqslant \Omega h^+(\tilde{x}) + \Omega h^-(\tilde{x})$.

Since $|h^+(x+t) - h^+(x)| \leqslant |h(x+t) - h(x)|$ and similarly with h^+ replaced by h^-, looking at the positive and negative parts reduces the $\Lambda^{p,\infty}_{\beta'}$ norm of a function. Since h^+ is nonnegative, the Harnack inequality for nonnegative α-harmonic functions shows that $u_{h^+}(z) \leqslant c_5 u_{h^+}(\tilde{x}, z_d)$ if $z \in \Gamma_{\tilde{x}}$, and similarly with h^+ replaced by h^-. Therefore by (4.7),

$$u_{h^+}(\tilde{x}, x_d) \leqslant c_6 c_4(a,b) Mh^+(\tilde{x}) + c_7 \varepsilon \|h^+\|_\infty$$

if $x_d < \gamma$, similarly with h^+ replaced by h^-, and therefore

$$\big|u_h(\tilde{x}, x_d)\big| \leqslant c_8 c_4(a,b) Mh(\tilde{x}) + c_9 \varepsilon \|h\|_\infty,$$

if $x_d < \gamma$. We conclude

$$\Omega h(\tilde{x}) \leqslant c_{10} c_4(a,b) Mh(\tilde{x}) + c_{11} \varepsilon \|h\|_\infty \leqslant c_{10} c_4(a,b) Mh(\tilde{x}) + c_{11} \eta.$$

Let $\zeta > 2c_{11}\eta$. Therefore,

$$\begin{aligned} \big|\big\{\tilde{x}\colon \Omega h(\tilde{x}) > \zeta\big\}\big| &\leqslant \left|\left\{\tilde{x}\colon Mh(\tilde{x}) > \frac{\zeta}{2c_{10}c_4(a,b)}\right\}\right| \leqslant \frac{2c_{10}c_4(a,b)}{\zeta} \|Mh\|_1 \\ &\leqslant \frac{c_{12}c_4(a,b)}{\zeta} J^{(d-1)(1-(1/p))} \|h\|_{\Lambda^{p,\infty}_{\beta'}} \\ &\leqslant \frac{c_{12}c_4(a,b)}{\zeta} J^{(d-1)(1-(1/p))} \delta \leqslant \frac{c_{12}}{\zeta} \eta, \end{aligned}$$

where we used Proposition 3.1 (but with β' instead of β). So

$$m\{\tilde{x}\colon \Omega f(\tilde{x}) > \zeta\} \leqslant \frac{c_{12}}{\zeta}\eta.$$

Since η can be chosen arbitrarily small, we get $\Omega f \leqslant \zeta$ almost everywhere. Now letting $\zeta \to 0$, we have our result. $\square$

We now prove Theorem 1.1.

Proof. Suppose that f is locally bounded, $p_0 > \frac{1}{1-\alpha/2}$, and $\beta p > 1$. Let $M > 0$. We will show nontangential convergence for $\tilde{x}$ in $B(0, M/2) \cap \partial H$. Since M is arbitrary, the theorem will follow. Let $\varphi \in C^\infty$ be a cut-off function such that $\varphi = 1$ on $B(0, M)$ and 0 on $B(0, 2M)^c$. Since f is locally bounded, $f\varphi$ is bounded and supported on $B(0, 2M)$. Since φ is smooth, then $f\varphi \in \Lambda_\beta^{p,\infty}(H^c)$, so by Proposition 4.2 the Fatou theorem holds for this function. Let $k(x) = f(x)(1 - \varphi(x)) \in L^{p_0}(\mathbb{R}^d)$ and note that $f = f\varphi + f(1 - \varphi)$ and hence $u_f(\tilde{x}, x_d) = u_{f\varphi}(\tilde{x}, x_d) + u_k(\tilde{x}, x_d)$. Thus it is enough to show for $x \in B(0, M/2) \cap \partial H$ that

$$\lim_{z\to\tilde{x},\ z\in\Gamma_{\tilde{x}}} u_k(z) = 0.$$

By writing $k = k^+ - k^-$, it is enough to consider the case where $k \geqslant 0$. Using the Harnack inequality for u_k as above, it is enough to show $u_k(\tilde{x}, x_d) \to 0$ as $x_d \to 0$.

For $x \in B(0, M/2) \cap H$ we have

$$\begin{aligned} u_k(\tilde{x}, x_d) &= c(\alpha, d) \int\limits_{B(0,M)^c} \left(\frac{x_d}{|z_d|}\right)^{\alpha/2} \frac{1}{|x - z|^d} k(z)\, dz \\ &\leqslant c(\alpha, d) \Biggl(\int\limits_{B(0,M)^c} \left(\frac{x_d}{|z_d|}\right)^{\alpha q_0/2} \frac{1}{|x - z|^{dq_0}}\, dz \Biggr)^{1/q_0} \left(\int k(z)^{p_0}\, dz \right)^{1/p_0} \end{aligned}$$

by Hölder's inequality, where $p_0^{-1} + q_0^{-1} = 1$. Note $\alpha q_0/2 < 1$. Since $|x - z| > |z|/2$, for $z \in H^c \cap B(0, M)^c$

$$\begin{aligned} \int\limits_{B(0,M)^c} \left(\frac{x_d}{|z_d|}\right)^{\alpha q_0/2} \frac{1}{|x - z|^{dq_0}}\, dz &\leqslant 2^{dq_0} \int\limits_{\{|z|>M,\ |z_d|>M\}} \left(\frac{x_d}{|z_d|}\right)^{\alpha q_0/2} \frac{1}{|z|^{dq_0}}\, dz \\ &\quad + 2^{dq_0} \int\limits_{\{|z|>M,\ |z_d|\leqslant M\}} \left(\frac{x_d}{|z_d|}\right)^{\alpha q_0/2} \frac{1}{|z|^{dq_0}}\, dz. \end{aligned}$$

We have

$$\begin{aligned} &\int\limits_{\{z\colon |z|>M,\ |z_d|\leqslant M\}} \left(\frac{x_d}{|z_d|}\right)^{\alpha q_0/2} \frac{1}{|z|^{dq_0}}\, dz \\ &\quad\leqslant \int_{-M}^{0} \left(\frac{x_d}{|z_d|}\right)^{\alpha q_0/2} dz_d \int\limits_{\{|\tilde{z}|>M\}} \frac{1}{|\tilde{z}|^{dq_0}}\, d\tilde{z}, \end{aligned} \tag{4.8}$$

and the right-hand side goes to 0 as $x_d \to 0$, since $\alpha q_0/2 < 1$. Also,

$$\int\limits_{\{z:\ |z|>M,\ |z_d|>M\}} \left(\frac{x_d}{|z_d|}\right)^{\alpha q_0/2} \frac{1}{|z|^{dq_0}}\,dz \leqslant \int\limits_{\{z:\ |z|>M,\ |z_d|>M\}} \left(\frac{x_d}{M}\right)^{\alpha q_0/2} \frac{1}{|z|^{dq_0}}\,dz, \tag{4.9}$$

and the right-hand side goes to 0 as $x_d \to 0$. Combining (4.8) and (4.9), $u_k(\tilde{x}, x_d) \to 0$ as $x_d \to 0$. □

Remark. The $\Lambda_\beta^{p,\infty}(\mathbb{R}^d)$ norm of f is defined by (1.2), but in the proof of Theorem 1.1 we used only the fact that

$$\sup_{|t|>0} \frac{\|f(x+t)-f(x)\|_p}{|t|^\beta}$$

is finite and that f is locally bounded. It is not necessary that f be in $L^p(\mathbb{R}^d)$ for p such that $\beta p > 1$, and hence we do not require that $p = p_0$.

5. Examples

It is easy to find nonnegative $f \in L^{p_1}$ with $p_1 = 1/(1-\alpha/2)$ such that $\int P_H((\tilde{0}, 1), y) \times f(y)\,dy = \infty$. By the Harnack inequality for nonnegative α-harmonic functions, u_f will be identically infinite, and so no Fatou theorem can hold. Therefore it is essential that the L^{P_0} condition be in Theorem 1.1.

Next we look at the locally bounded condition. Let $g(z) = \log^+(1/|z_d|)$ for $z \in H^c$ and let f be g multiplied by a smooth cut-off function that is 1 on $B(0, M)$ for some M. It is easy to check that f is in L^p for all p, so that u_f is well defined. It is also easy to check, using Proposition 2.1, that $u_f(x)$ tends to infinity as $x_d \to 0$ as long as $\tilde{x}$ is in the support of f.

We now want to show that the condition $\beta p > 1$ is sharp, that is, that Theorem 1.2 holds.

Proof. Let $L > 1$ be a number to be chosen in a moment. Define $h_L(s)$ to be 1 if $s \in [-L^{6n+1}, -L^{6n-1}]$ for some integer n and $h_L(s)$ to be 0 if $s \in [-L^{6n+4}, -L^{6n+2}]$ for some integer n. Define h_L to be linear on each interval $(-L^{6n+2}, -L^{6n+1})$ and $(-L^{6n+5}, -L^{6n+4})$. Note that $h_L(L^{-3}s) = 1 - h_L(s)$ for $s < 0$. Define $g_L(x) = h_L(x_d)$, where $x = (\tilde{x}, x_d)$. So we have

$$\begin{aligned} u_{g_L}(\tilde{x}, L^{-3}x_d) &= c(\alpha, d) \int\limits_{H^c} \left(\frac{L^{-3}x_d}{|y_d|}\right)^{\alpha/2} \frac{g_L(y)}{|(\tilde{x}, L^{-3}x_d) - y|^d}\,dy \\ &= c(\alpha, d) \int\limits_{H^c} \left(\frac{x_d}{|y_d|}\right)^{\alpha/2} \frac{1 - g_L(y)}{|x-y|^d}\,dy \\ &= u_{1-g_L}(x) = 1 - u_{g_L}(x). \end{aligned}$$

As $L \to \infty$, we have $g_L \to \chi_{H^c}$ and $u_{g_L}(\tilde{x}, 1) \to 1$ by dominated convergence. We now choose L sufficiently large so that $u_{g_L}(\tilde{x}, 1) = u_{g_L}(\tilde{0}, 1) > \frac{1}{2}$. Then $1 - u_{g_L}(\tilde{x}, 1) \neq u_{g_L}(\tilde{x}, 1)$, and we conclude that $u_{g_L}(\tilde{x}, L^{-3n})$ diverges as $n \to \infty$.

Let f be equal to $g_L \varphi$, where φ is a smooth cut-off function that is equal to 1 on $B(0, M)$ and is supported in $B(0, 2M)$. Since f is bounded, we can argue as in the proof of Theorem 1.1 that $u_f - u_{g_L}$ converges nontangentially for $\tilde{x}$ in $\partial H \cap B(0, M/2)$. Therefore u_f diverges on this set. Define f on H by reflecting over ∂H. It remains to show that $f \in \Lambda_\beta^{p,\infty}(\mathbb{R}^d)$ when $\beta \in (0, 1)$, $p > 1$, and $\beta p = 1$. Since f is bounded with compact support, it suffices to show

$$\left(\int \left| f(x+t) - f(x) \right|^p dx \right)^{1/p} \leqslant c_1 t^\beta .$$

Observe that

$$\begin{cases} |g_L(x+t) - g_L(x)| \leqslant 2, & \text{if } |x_d| \leqslant 4t, \\ |g_L(x+t) - g_L(x)| \leqslant c_2 L^{-3n}, & \text{if } L^{3n} t \leqslant |x_d| \leqslant L^{3n+3} t,\ n \geqslant 0. \end{cases}$$

Choose $N_0 < 0$ so that $L^{-3N_0} > 2M$. Since f has compact support, then

$$\begin{aligned} &\int \left| f(x+t) - f(x) \right|^p dx \\ &\quad \leqslant c_3 \int\limits_{|x_d| \leqslant 4t} 2^p \, dx_d + c_3 \sum_{n=N_0}^{\infty} \int\limits_{L^{-3n} t}^{L^{-3n+3} t} \left| h_L(x+t) - h_L(x) \right|^p dx_d \\ &\quad \leqslant c_4 t + c_5 \sum_{n=N_0}^{\infty} \left(c_2 L^{-3n} \right)^p L^{3n} \left(L^3 - 1 \right) t \leqslant c_6 t, \end{aligned}$$

for some constant c_6, since $p > 1$. (c_6 depends on L and M.) Therefore

$$\left(\int \left| f(x+t) - f(x) \right|^p dx \right)^{1/p} \leqslant c_7 t^{1/p} = c_7 t^\beta .$$

This means $f \in \Lambda_\beta^{p,\infty}(\mathbb{R}^d)$, and the proof is complete. □

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